

No-hole λ - $L(k, k-1, \dots, 2, 1)$ -labeling for Square Grid

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Abstract

Given a fixed $k \in \mathbb{Z}^+$ and $\lambda \in \mathbb{Z}^+$, the objective of a λ - $L(k, k-1, \dots, 2, 1)$ -labeling of a graph G is to assign non-negative integers (known as labels) from the set $\{0, \dots, \lambda-1\}$ to the vertices of G such that the adjacent vertices receive values which differ by at least k , vertices connected by a path of length two receive values which differ by at least $k-1$, and so on. The vertices which are at least $k+1$ distance apart can receive the same label. A λ - $L(k, k-1, \dots, 2, 1)$ -labeling of a graph G is said to be a no-hole labeling if all the labels between 0 and $\lambda-1$ are used at least once. The ratio between the upper bound and the lower bound of a λ - $L(k, k-1, \dots, 2, 1)$ -labeling is known as the approximation ratio. In this paper a lower bound of the problem is presented and a formula is proposed which yields a no-hole λ - $L(k, k-1, \dots, 2, 1)$ -labeling of square grid with at most $\frac{9}{8}$ approximation ratio.

1 Introduction

The *frequency assignment problem* (FAP) is a problem of assigning frequencies to different radio transmitters so that no interference occurs [1]. This problem is also known as *channel assignment problem* (CAP) [2, 3]. Frequencies are assigned to different radio transmitters in such a way that comparatively close transmitters receive frequencies with more gap than the transmitters which are significantly apart from each other. Motivated by this problem of assigning frequencies to different transmitters, Yeah [4] and after that Griggs and Yeh [5] proposed an $L(2, 1)$ -labeling for a simple graph. An $L(2, 1)$ -labeling of a graph G is a mapping $f : V(G) \rightarrow \mathbb{Z}^+$ such that $|f(u) - f(v)| \geq 2$, when $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$, when $d(u, v) = 2$, where $d(u, v)$ denotes the minimum path distance between the two vertices $u, v \in V$ [5, 6, 7, 8, 9]. If the distance between two vertices is greater than 2, then they can be labeled with the same label. The smallest λ for which a valid $L(2, 1)$ -labeling exists, is known as the *labeling number*, and is denoted as λ_G [9]. Chang et al. [10] considered a bit more general $L(i, j)$ -labeling problem of a graph G and found the corresponding labeling number $\lambda_{i,j}(G)$. A labeling of a graph G is said to be a *no-hole labeling* if all the labels between 0 and $\lambda-1$ are used. If there exists a labeling of a graph G with λ as its labeling number, then the number of labels not used in the

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labeling of the graph is known as the *hole index* of corresponding labeling [11]. A detailed survey on k - $L(d_1, d_2)$ -labeling problem for a graph can be found in [12]. Instead of $L(2, 1)$ -labeling one can consider $L(3, 2, 1)$ -labeling, and more generally an $L(k, k-1, \dots, 1)$ -labeling. Nandi et al. [13] considered an $L(k, k-1, \dots, 1)$ -labeling for a triangular lattice.

In this paper $L(k, k-1, \dots, 2, 1)$ -labeling for a square grid is considered. The definition of the problem is given in Section 2. The lower bound on the value of the labeling number λ_k for a square grid is derived in Section 3. In Section 4, a formula is given that yields a label for any vertex of an infinite square grid for arbitrary values of k . The upper bound on λ_k is given in Section 4.1. The correctness proof of the proposed formula is given Section 4.2. In Section 4.3 we prove that the proposed labeling formula gives a no-hole labeling. The approximation ratio of the proposed formula is derived in Section 4.4 based on the proposed lower bound. Finally the paper is concluded in Section 5.

2 Problem Definition

Let $G = (V, E)$ be a graph with a set of vertices V and a set of edges E , and let $d(u, v)$ denote the shortest distance between vertices $u, v \in V$. Given a fixed $k \in \mathbb{Z}^+$ and $\lambda \in \mathbb{Z}^+$, a λ - $L(k, k-1, \dots, 2, 1)$ -labeling of the graph is a mapping $f : V \rightarrow \{0, \dots, \lambda-1\}$ such that the following inequalities are satisfied:

$$|f(x) - f(y)| \geq \begin{cases} k & : d(x, y) = 1 \\ k-1 & : d(x, y) = 2 \\ \vdots & \\ 1 & : d(x, y) = k, \end{cases}$$

which can be written more compactly as

$$|f(x) - f(y)| \geq k + 1 - d(x, y) \text{ for } x \neq y. \quad (*)$$

We shall call any function $f : V \rightarrow \mathbb{Z}$ satisfying the inequality a *labelling function*.

If the distance between two vertices is at least $k+1$, the same label can be used for both of them. This minimum distance is known as the *reuse distance* [13]. The $L(k, k-1, \dots, 2, 1)$ -labeling number, λ_k for the graph is the minimum λ for which a valid λ - $L(k, k-1, \dots, 2, 1)$ -labeling for the graph exists. Hence, the objective of this problem is to minimize λ_k .

We consider an infinite planar square grid $G = (V, E)$ with the set of vertices $V = \mathbb{Z} \times \mathbb{Z}$ and the set of edges $E = \{\{u, v\} : u = (u_1, u_2), v = (v_1, v_2), \text{ and either } |u_1 - v_1| = 1, u_2 = v_2 \text{ or } u_1 = v_1, |u_2 - v_2| = 1\}\}$. It will be called ‘the square grid’ in the sequel. The distance between u and v used in the sequel is the *Manhattan distance*: $d(u, v) = |u_1 - v_1| + |u_2 - v_2|$.

3 Lower Bound on λ_k

Theorem 1.

$$\lambda_k \geq \begin{cases} \frac{2}{3}p(p+1)(2p+1) + 2 & \text{if } k = 2p \text{ is even,} \\ \frac{2}{3}p(p+1)(2p+3) + 2 & \text{if } k = 2p+1 \text{ is odd,} \end{cases}$$

where p is any non-negative integer.

Proof. We start with the case of even $k = 2p$. We shall write B_m for the ball $\{u \in V : d(0, u) \leq m\}$, and S_m for the sphere $\{u \in V : d(0, u) = m\}$ (here $0 = (0, 0)$). Note that there is just one point in S_0 and $4m$ points in S_m for $m > 0$ (See Figure 1). It is easy to calculate that there are exactly $1 + 4 + \dots + 4m = 2m^2 + 2m + 1$ points in B_m . To obtain a lower bound on the $L(k, k-1, \dots, 2, 1)$ -labeling number, we identify the smallest interval containing all integers needed to label the vertices in the ball B_p . To this aim, we use a labeling function $f : V \rightarrow \mathbb{Z}$. It is clear that $\lambda_k \geq \max f(B_p) - \min f(B_p) + 1$.

Let us put all the values of the function f on B_p in increasing order: $z_0 < z_1 < \dots < z_n$. We have $\lambda_k \geq z_n - z_0 + 1$. Note that because of (*), the function f is injective on B_p , hence $n = 2p^2 + 2p$ is one less than the number of points in B_p . Let $u_i = f^{-1}(z_i)$ and let q, r be such that $u_0 \in S_q, u_n \in S_r$.

The first method of obtaining the lower bound is a formalization of that used by Nandi et al. [13]. According to (*), $z_{i+1} - z_i \geq 2p+1 - \max\{d(u_i, v) : v \in B_p \setminus \{u_i\}\}$. If $u_i \in S_m$, then $\max\{d(u_i, v) : v \in B_p \setminus \{u_i\}\} = m + p$, hence $z_{i+1} - z_i \geq p + 1 - m$. Considering z_i for $i = 0, 1, \dots, n-1$, we can already estimate that

$$z_n - z_0 = (z_1 - z_0) + \dots + (z_n - z_{n-1}) \geq |S_p| + 2|S_{p-1}| + \dots + p|S_1| + (p+1)|S_0| - (p+1-r).$$

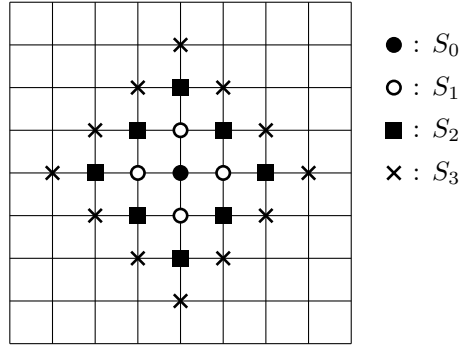


Figure 1: S_m when $m = 0, 1, 2, 3$.

Let us call the number on the RHS of the inequality c_p . Now, if a point u_i is such that $i < n$ and $u_{i+1} \in B_{p-1}$, then $z_{i+1} - z_i \geq 2p + 1 - \max\{d(u_i, v) : v \in B_{p-1} \setminus \{u_i\}\} = p + 2 - m$ (instead of $p + 1 - m$). There are at least $|B_{p-1}|$ points like this if $q = p$, and $|B_{p-1}| - 1$ if $q \neq p$, and the RHS of the inequality above can be increased by the amount. Continuing further in this manner, we get

$$\begin{aligned} z_n - z_0 &\geq c_p + (|B_{p-1}| - 1) + \dots + (|B_q| - 1) + |B_{q-1}| + \dots + |B_0| \\ &= c_p + |S_{p-1}| + 2|S_{p-2}| + \dots + (p-1)|S_1| + p|S_0| - (p-q) \\ &= 4\left(\sum_{m=1}^p m(p+1-m) + \sum_{m=1}^{p-1} m(p-m)\right) + (r+q). \end{aligned}$$

Using

$$1 \cdot p + 2 \cdot (p-1) + \dots + (p-1) \cdot 2 + p \cdot 1 = p(p+1)(p+2)/6,$$

and the fact that $r+q$ is at least 1, which happens if $p, q \in \{0, 1\}$ (note that they must be different, since there is only one point in S_0), we easily get $\lambda_k \geq \frac{2}{3}p(p+1)(2p+1) + 2$. Now, if $k = 2p+1$ is odd, each of the $2p^2 + 2p$ summands $z_1 - z_0, z_2 - z_1, \dots, z_n - z_{n-1}$ is larger by one, hence $\lambda_k \geq \frac{2}{3}p(p+1)(2p+3) + 2$. A better estimate can be obtained by considering the set $T_0 = \{(0,0), (0,1)\}$ and, for $m > 0$, the sets $T_m = \{u \in \mathbb{Z} \times \mathbb{Z} : d(u, T_0) = m\}$ (see Figure 2). This, however, does not change the asymptotic behavior of λ_k .

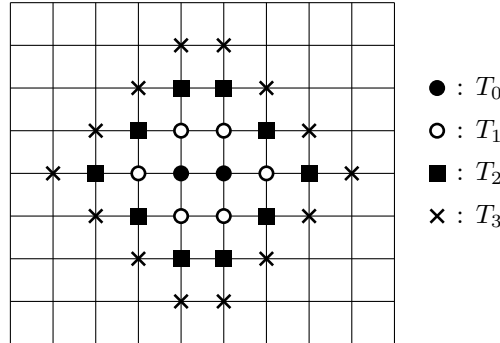


Figure 2: T_m when $m = 0, 1, 2, 3$.

□

4 Proposed Formula

In this section a formula is given to find the label of any vertex of a square grid under $L(k, k-1, \dots, 2, 1)$ -labeling for general k . Let the label assigned to the vertex $v(x, y)$ is denoted by $L(x, y)$. Formula 1 gives the definition of $L(x, y)$.

Formula 1.

$$L(x, y) = \begin{cases} [(2p+3)x + (3p^2 + 7p + 5)y] \bmod \frac{1}{2}(p+1)(3p^2 + 5p + 4) & \text{if } k = 2p+1 \text{ and } p(\geq 1) \text{ is odd;} \\ [(2p+3)x + (3p^2 + 6p + 3)y] \bmod \frac{1}{2}(3p^3 + 8p^2 + 8p + 4) & \text{if } k = 2p+1 \text{ and } p(\geq 0) \text{ is even;} \\ [(2p+1)x + (3p^2 + 4p + 2)y] \bmod \frac{1}{2}(3p^3 + 5p^2 + 5p + 1) & \text{if } k = 2p \text{ and } p(\geq 3) \text{ is odd;} \\ [(2p+1)x + (3p^2 + 3p + 1)y] \bmod \frac{1}{2}p(3p^2 + 5p + 4) & \text{if } k = 2p \text{ and } p(\geq 2) \text{ is even.} \end{cases}$$

Note that many correct labellings may exist when the coefficients of x and y are restricted to be co-prime. If this restriction is removed then correct labellings also exist with reduced λ_k . Thus we have considered all possible combinations of the coefficients for x and y at the time of designing Formula 1 for finding a labeling with the minimum λ_k . The assignment of labeling for $k = 7$ is shown in Figure 3 for some vertices.

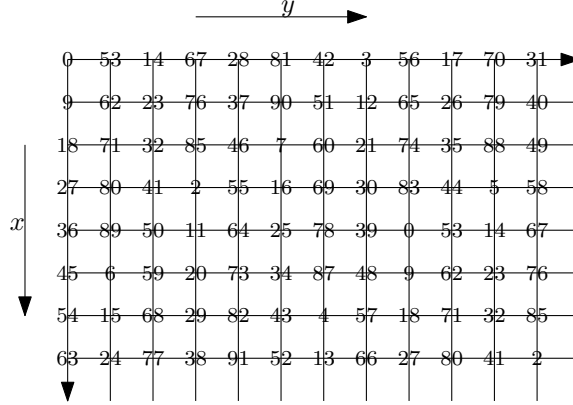


Figure 3: Assignment of labeling for $k = 7$

4.1 Upper Bound on λ_k

The upper bound on λ_k can be derived easily from Formula 1 and Lemma 1 directly follows from this.

Lemma 1.

$$\lambda_k \leq \begin{cases} \frac{1}{2}(p+1)(3p^2+5p+4) & \text{if } k = 2p+1 \text{ and } p(\geq 1) \text{ is odd;} \\ \frac{1}{2}(3p^3+8p^2+8p+4) & \text{if } k = 2p+1 \text{ and } p(\geq 0) \text{ is even;} \\ \frac{1}{2}(3p^3+5p^2+5p+1) & \text{if } k = 2p \text{ and } p(\geq 3) \text{ is odd;} \\ \frac{1}{2}p(3p^2+5p+4) & \text{if } k = 2p \text{ and } p(\geq 2) \text{ is even.} \end{cases}$$

4.2 Correctness Proof of the Proposed Formula

Formula 1 is said to be correct if and only if the inequality constraints of the problem mentioned in Section 2 is satisfied. We prove Theorem 2 to show the correctness of Formula 1. Lemma 2 is needed to prove Theorem 2.

Lemma 2. Let $a, b, c \in \mathbb{Z}^+$ and $L(x, y) = (ax + by) \bmod c$. Now for any $x_1, y_1, x_2, y_2 \in \mathbb{Z}$, if $L(x_1, y_1) > L(x_2, y_2)$ then $|L(x_1, y_1) - L(x_2, y_2)| = L(x_1 - x_2, y_1 - y_2)$.

Proof. Clearly $0 \leq L(x, y) < c$ for any $x, y \in \mathbb{Z}$. Hence, $0 \leq |L(x_1, y_1) - L(x_2, y_2)| < c$. Again, for any $A, B \in \mathbb{Z}$, $(A \bmod c - B \bmod c) \bmod c = (A - B) \bmod c$. Put $A = ax_1 + by_1$ and $B = ax_2 + by_2$. Then $|L(x_1, y_1) - L(x_2, y_2)| = A \bmod c - B \bmod c = (A \bmod c - B \bmod c) \bmod c = (A - B) \bmod c = L(x_1 - x_2, y_1 - y_2)$. \square

Theorem 2. If $|x_1 - x_2| + |y_1 - y_2| = r$, then $|L(x_1, y_1) - L(x_2, y_2)| \geq k + 1 - r$, where $0 < r \leq k + 1$ and $L(x, y)$ is defined by Formula 1.

Proof. We prove Theorem 2 when $L(x, y) = [(2p+3)x + (3p^2+7p+5)y] \bmod \frac{1}{2}(p+1)(3p^2+5p+4)$ and $k = 2p+1$, $p(\geq 3)$ is odd. We prove the correctness for $p = 1$ separately. The correctness of Formula 1 can be proved for other values of k similarly.

We can change the order of (x_1, y_1) and (x_2, y_2) in such a way that $L(x_1, y_1) \geq L(x_2, y_2)$, since exchanging indices 1 and 2 does not change r . By Lemma 2 we have to show that for $x, y \in \mathbb{Z}$ with $|x| + |y| = r$, $L(x, y) \geq k + 1 - r$. Note that the inequality is always satisfied for $r = k + 1$. Hence, we can assume $0 < r < k + 1$.

Put $a = 2p + 3, b = 3p^2 + 7p + 5$ and $c = \frac{p+1}{2}(3p^2 + 5p + 4)$. Note that $|ax + by| < 5c$ for any x, y with $|x| + |y| = r$.

Case-I Assume that $ct \leq by \leq ax + by < c(t + 1)$ for some $t \in [-5, 4] \cap \mathbb{Z}$. Then

$$(ax + by) \bmod c = ax + by - ct \geq ax > 2p + 2.$$

(Since $x > 0$, $ax \geq a = 2p + 3$.) Hence, $L(x, y) > 2p + 2 = k + 1 \geq k + 1 - r$.

Case-II Assume that $x = 0$. Let $Y_t = \{y : ct \leq by < c(t+1)\}$ and $y_t = \min(Y_t)$, $t \in [-5, 4] \cap \mathbb{Z}$ for $|y_t| \leq k$. Note that $b > 0$, so that whenever $L(x, y_t) \geq k+1$, also $\forall y \in Y_t$, $L(x, y) \geq k+1$. Since $y \neq 0$ (we already have $x = 0$), so $y_0 = 1$ and $by_0 \bmod c = b > 2p+2 = k+1$. Hence, we need only consider $t \neq 0$. Put $d = \frac{2p^2+3p+1}{6p^2+14p+10} = \frac{p+1}{2} \frac{2p+1}{b}$. Note that for each odd $p \neq 1$, $\frac{1}{4} < d < \frac{1}{3}$. Now $y_t \geq ct/b = t(\frac{p+1}{2} - d)$, so that $y_t = t\frac{p+1}{2} + e$, where

$$e = \begin{cases} 0 & \text{if } t = 1, 2 \text{ or } 3; \\ -1 & \text{if } t = 4; \\ 1 & \text{if } t = -1, -2 \text{ or } -3; \\ 2 & \text{if } t = -4 \text{ or } t = -5. \end{cases}$$

We have $L(0, y_t) = by_t - ct = t(b\frac{p+1}{2} - c) + be = t(2p^2+3p+1)/2 + be$. The inequality $L(0, y_t) \geq 2p+2$ is obviously true if t is positive and $e = 0$. If $t = 4$, we have $L(0, y_t) = 2(2p^2+3p+1) - b = p^2 - p - 3 \geq 2p+2$ for odd $p \geq 5$, and $L(0, y_t) \geq k+1 - r$ for $p = 3$. For $t = -1, -2$ or -3 , it is enough to check the “worst” case, namely $t = -3$, which yields $L(0, y_t) = (5p+7)/2 \geq 2p+2$. Again, we can omit $t = -4$ and check that for $t = -5$ we get $L(0, y_t) = (2p^2+13p+15)/2 \geq 2p+2$.

Case-III Assume that $by < ct \leq ax + by$. Note that then $c(t-1) < by < ct \leq ax + by < c(t+1)$. We will show that there exist at most two y 's satisfying the inequality. Let $y_t = \max\{y : by < ct \wedge (\exists x : ct \leq ax + by)\}$. Thus $by_t < ct \leq ax + by_t$ for some x . Suppose $b(y_t - 2) < ct \leq ax + b(y_t - 2)$ for some x . Then $ax + b(y_t - 1) = (ax - 2b) + by_t \geq ct > by_t$. But $ax - 2b \leq a(2p+2) - 2b = 2[(p+1)(2p+3) - (3p^2+7p+5)] = 2(-p^2 - 2p - 2) < 0$, which is a contradiction. If we find $x_t = \min\{x : by_t < ct \leq ax + by_t\}$ and $x'_t = \min\{x : b(y_t - 1) < ct \leq ax + b(y_t - 1)\}$ and if $|x_t| + |y_t| < 2p+2$ (similarly $|x'_t| + |y_t| < 2p+2$), then it is enough to check that $L(x_t, y_t) \geq k+1 - r$ and $L(x'_t, y_t - 1) \geq k+1 - r$.

Put $d = \frac{2p^2+3p+1}{6p^2+14p+10} = \frac{p+1}{2} \frac{2p+1}{b}$. Note that for each odd $p \neq 1$, $\frac{1}{4} < d < \frac{1}{3}$. Now $y_t < ct/b = t(\frac{p+1}{2} - d)$, so that $y_t = t\frac{p+1}{2} + e$, where

$$e = \begin{cases} -1 & \text{if } t = 1, 2 \text{ or } 3; \\ -2 & \text{if } t = 4; \\ 0 & \text{if } t = -1, -2, -3 \text{ or } -4; \\ 1 & \text{if } t = -5. \end{cases}$$

Using $ct \leq ax_t + by_t \Rightarrow x_t \geq \frac{ct - by_t}{a}$, and $L(x_t, y_t) = ax_t + by_t - ct$, the Table 1 is constructed. Whenever $|y_t|$, $|x_t|$ or r is at least $2p+2$, there is no need for further calculation.

Table 1

t	y_t	x_t	$r = x_t + y_t $	$k+1-r$	$L(x_t, y_t)$
1	$\frac{p-1}{2}$	$(p+2)$	$\frac{3}{2}(p+1)$	$\frac{1}{2}(p-3)$	$\frac{3}{2}(p+1)$
2	p	$\frac{(p+3)}{2}$	$\frac{3}{2}(p+1)$	$\frac{1}{2}(p-3)$	$\frac{1}{2}(p+1)$
3	$\frac{(3p+1)}{2}$	2	$\frac{1}{2}(3p+5)$	$\frac{1}{2}(p-1)$	$\frac{1}{2}(3p+5)$
4	$2p$	$(p+1)$	$3p+1$	—	—
-1	$-\frac{(p+1)}{2}$	$\frac{(p+1)}{2}$	$p+1$	$p+1$	$p+1$
-2	$-(p+1)$	$(p+1)$	$2(p+1)$	—	—
-3	$-\frac{3(p+1)}{2}$	$(3p+1)$	$\frac{1}{2}(9p+5)$	—	—
-4	$-2(p+1)$	—	—	—	—
-5	$-\frac{(5p+3)}{2}$	—	—	—	—

Using $ct \leq ax'_t + b(y_t - 1) \Rightarrow x'_t \geq \frac{ct - b(y_t - 1)}{a}$, and $L(x'_t, y_t - 1) = ax'_t + b(y_t - 1) - ct$, the Table 2 is constructed. In this case also whenever $|y_t - 1|$, $|x'_t|$ or r is at least $2p+2$, there is no need for further calculation.

Case-IV Assume that $ax + by < ct \leq by$, where $t \in [-4, 4] \cap \mathbb{Z}$. Then $c(t-1) < ax + by < ct \leq by < c(t+1)$ and $ax + by \geq ax + ct = c(t-1) + (ax + c)$. Hence, $L(x, y) = (ax + by) \bmod c = ax + c$.

Since $ax \geq a(-2p-2) = -2(2p+3)(p+1)$, we have

$$L(x, y) = \frac{(p+1)(3p^2+5p+4)}{2} - 2(2p+3)(p+1) = \frac{3}{2}p^3 - \frac{11}{2}p - 4 \geq 2p+2, \text{ for } p \geq 3.$$

Therefore, for $p \geq 3$, $L(x, y) \geq k+1 - r$.

Case-V Assume that $x < 0$, $ax + by \geq ct$ and $by < c(t+1)$.

Table 2

t	$y_t - 1$	x'_t	$r = x'_t + y_t - 1 $	$k + 1 - r$	$L(x'_t, y_t - 1)$
1	$\frac{p-3}{2}$	$\frac{(5p+7)}{2}$	$3p + 2$	—	—
2	$(p - 1)$	$2p + 3$	—	—	—
3	$\frac{(3p-1)}{2}$	$\begin{cases} \frac{3(p+1)}{2}, & \text{if } p = 3, 5 \\ \frac{3p+1}{2}, & \text{if } p(\geq 7) \end{cases}$	—	—	—
4	$2p - 1$	$\frac{(5p+9)}{2}$	—	—	—
-1	$-\frac{(p+3)}{2}$	$2p + 2$	—	—	—
-2	$-(p + 2)$	$\frac{(5p+3)}{2}$	—	—	—
-3	$-\frac{(3p+5)}{2}$	$(3p + 2)$	—	—	—
-4	$-(2p + 3)$	—	—	—	—
-5	$-\frac{5(p+1)}{2}$	—	—	—	—

Let $Y_t = \{y : \exists x \text{ s.t. } ct \leq ax + by < by < c(t + 1)\}$. Then it is enough to check the inequality for $y_t = \min(Y_t)$ and for $y_t + 1$, and for them we should check if for $x_t = \min\{x : ct \leq ax + by_t < by_t < c(t + 1)\}$ and $x'_t = \min\{x : ct \leq ax + b(y_t + 1) < b(y_t + 1) < c(t + 1)\}$.

Thus we need to check $L(x_t, y_t) \geq k + 1 - r$ and $L(x'_t, y_t + 1) \geq k + 1 - r$.

Using $by_t < c(t + 1)$, Table 3 is constructed.

Table 3

t	1	2	3	4	-1	-2	-3	-4	-5
y_t	p	$\frac{(3p+1)}{2}$	$2p + 1$	$\frac{(5p-1)}{2}$	-1	$-\frac{(p+1)}{2}$	$-(p + 1)$	$-\frac{3(p+1)}{2}$	$-(2p + 1)$

If we calculate the values of x_t and x'_t from $ct \leq ax_t + by_t$ and $ct \leq ax'_t + b(y_t + 1)$ respectively, then x_t and x'_t are always greater than $2p + 2$. This completes the proof for $p \geq 3$.

•Note that when $p = 1$, $k = 3$ and $L(x, y) = (5x + 15y) \bmod 12$. When $p = 1$, we just need to consider different values of x and y such that $x \in \{-3, -2, -1\}$ and $y \in \{-3, -2, -1, 0, 1, 2, 3\}$. Clearly when $(x, y) \in \{(-3, -3), (-3, -2), (-3, -1), (-3, 0), (-3, 1), (-3, 2), (-3, 3), (-2, -3), (-2, -2), (-2, -1), (-2, 0), (-2, 1), (-2, 2), (-2, 3), (-1, -3), (-1, -2), (-1, -1), (-1, 0), (-1, 1), (-1, 2), (-1, 3)\}$, we don't to check anything because $r = |x| + |y| \geq 4$. When $(x, y) = (-3, 0)$, $L(x, y) = \{5 \times (-3) + 15 \times 0\} \bmod 12 = 9$ and $k + 1 - r = 3 + 1 - 3 = 1$. Similarly, when $(x, y) \in \{(-2, -1), (-2, 0), (-2, 1), (-1, -2), (-1, -1), (-1, 0), (-1, 1), (-1, 2)\}$, $L(x, y) \geq (k + 1 - r)$.

Hence always $L(x, y) \geq (k + 1 - r)$.

□

4.3 No-hole Labeling Proof

Lemma 3. *Formula 1 gives no-hole labeling.*

Proof. Formula 1 is of the form $(ax + by) \bmod c$. We shall show that it is enough to check that $\gcd(a, b, c)$ is 1. In fact, let $m = \gcd(a, b)$ and denote by (m) the principal ideal in \mathbb{Z} generated by m . It is well known (and easy to see) that the set $\{ax + by : x, y \in \mathbb{Z}\}$ equals (m) . Now, if $\gcd(m, c) = \gcd(a, b, c) = 1$, then $mu + cv = 1$ for some $u, v \in \mathbb{Z}$. If $k \in \{0, 1, \dots, c - 1\}$, then $kmu + kcv = k$, so that $kmu \equiv k \bmod c$. But $kmu \in (m)$, which means that for some $x, y \in \mathbb{Z}$, $(ax + by) \bmod c = k$, and all integer values from 0 up to $c - 1$ are attained. □

4.4 Approximation Ratio

The approximation ratio is the ratio between the upper bound (UB) given in Lemma 1 and the lower bound (LB) given in Theorem 1. Note that for all the cases mentioned in Formula 1, $\lim_{p \rightarrow \infty} \frac{UB}{LB} = \frac{9}{8}$. Therefore, Formula 1 gives the λ - $L(k, k - 1, \dots, 2, 1)$ -labeling of square grid with at most $\frac{9}{8}$ approximation ratio.

5 Conclusion

In this paper λ - $L(k, k-1, \dots, 2, 1)$ -labeling for square grid is proposed and the lower bound on the $L(k, k-1, \dots, 2, 1)$ -labeling number, λ_k is computed. A formula for a no-hole λ - $L(k, k-1, \dots, 2, 1)$ -labeling of square grid is given in this paper with at most $\frac{9}{8}$ approximation ratio. The correctness proof of the proposed formula is given and it is also proved that the proposed formula gives no-hole labeling.

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